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On the Bounds of Laplacian Energy for Degree Product Adjacency Matrix of Regular Graph

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Abstract

In this article, lower bounds for the Laplacian energy of degree product adjacency matrix of regular graphs *G* were established.

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1. Introduction

Let G be a simple, connected graph with n-vertices and m-edges. d_i is the degree of the vertex v_i, where d_i is the number of edges incident to the vertex v_i. The graph G is a regular graph, where all its vertices are equal to degree r. For undefined terminologies we refer [7].

A molecular graph is a graph in which the vertices correspond to the atoms and edges corresponds to the bonds. Chemical graph theory is a branch of mathematical chemistry which has an important effect on the development of the chemical sciences.

The adjacency matrix $A(G)$ of a graph G will be (0, 1) matrix and is defined as

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the eigen values of $A(G)$. Then the energy of a graph G is defined as the sum of absolute values of the eigen values of adjacent matrix of graph G. This concept was introduced by I. Gutman in [4].

$$E_A(G) = \sum_{i=1}^{n} |\lambda_i|$$

The degree product adjacency energy [$E_{DPA}(G)$] is defined as follows [8],

The $DPA(G)$ is the degree product adjacency matrix and is defined as,

$$d_{ij} = \begin{cases} d_i d_j & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

The degree product adjacency matrix $DPA(G)$ is a real symmetric matrix and its eigen values are $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$. The order of eigen values be $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \ldots \geq \alpha_n$. The similar way of adjacency energy, the degree product adjacency energy of a graph defined as,

$$E_{DPA}(G) = \sum_{i=1}^{n} |\alpha_i| \qquad (1)$$

Let $D(G) = [d_{ij}]$ be a diagonal matrix and is defined as,

$$d_{ij} = \begin{cases} \deg(v_i) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The Laplacian matrix of a graph G is defined as $L(G) = D(G) - A(G)$. The eigen values $\mu_1 \geq \mu_2 \geq \mu_3 \geq \dots \geq \mu_n$ are obtained from the matrix $L(G)$.

The Laplacian energy of G is defined as [5], $LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$, where n and m are the number of vertices and edges respectively.

Motivated by the work of [5], we introduced the concept of Laplacian energy for the degree product adjacency matrix, which is defined as follows

Let G be a simple, connected graph with n-vertices $v_1, v_2, \dots, v_n$ and d_i be the degree of the vertex v_i, $\forall i = 1, 2, \dots, n$. Then the Laplacian degree product adjacency matrix $[L_{DPA}(G)]$ of a graph G is $L_{DPA}(G) = D(G) - DPA(G)$. The eigen values $\gamma_1 \geq \gamma_2 \geq \gamma_3 \geq \dots \geq \gamma_n$ are obtain from the matrix $L_{DPA}(G)$.

Analogous to the $LE(G)$, the $LE_{DPA}(G)$ of a graph G defined and is denoted as follows

$$LE_{DPA}(G) = \sum_{i=1}^{n} |\beta_i| \tag{2}$$

where $\beta_i = \gamma_i - \frac{2m}{n}$, $\forall i = 1, 2, \dots, n$.

where n and m are the number of vertices and edges of graph G respectively.

The Cauchy-Schwarz inequality [1] states that if $(a_1, a_2, a_3, \dots, a_n)$ and $(b_1, b_2, b_3, \dots, b_n)$ are real n-vectors then,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right)$$

In this article, we establish the results on bound for the largest eigen value of $L_{DPA}(G)$ and also obtain the lower bounds for the Laplacian energy of degree product adjacency matrix of a regular graph G.

2. Results

To present the complete results, some important theorems which are used throught out the paper are mentioned below.

Theorem 2.1. [11] Suppose a_i and b_i, $1 \leq i \leq n$ are positive real numbers, then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2 \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

Where

$$M_1 = \max_{1 \leq i \leq n}(a_i); M_2 = \max_{1 \leq i \leq n}(b_i); m_1 = \min_{1 \leq i \leq n}(a_i); m_2 = \min_{1 \leq i \leq n}(b_i)$$

Theorem 2.2. [10] Let a_i and b_i, $1 \leq i \leq n$ are nonnegative real numbers, then

$$\sum_{i=1}^{n} {a_i}^2 \sum_{i=1}^{n} {b_i}^2 - \left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \frac{n^2}{4} (M_1 M_2 - m_1 m_2)^2$$

where M_1M_2 and m_1m_2 are defined similarly to Theorem 2.1.

Theorem 2.3. [2] Suppose a_i and b_i, $1\leq i \leq n$ are positive real numbers, then

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \mu(n)(A-a)(B-b)$$

where a, b, A and B are real constants, that for each i, $1\leq i \leq n$, $a\leq a_i \leq A$ and $b\leq b_i \leq B$ Further, $\mu(n) = n\left\lfloor \frac{n}{2} \right\rfloor \left(1 - \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor \right)$.

Theorem 2.4. [3] Let a_i and b_i, $1\leq i \leq n$ are nonnegative real numbers, then

$$\sum_{i=1}^{n} {b_i}^2 + rR\sum_{i=1}^{n} {a_i}^2 \leq (r+R)\left(\sum_{i=1}^{n} a_i b_i \right)$$

where r and R are real constants. So that for each i, $1\leq i \leq n$ holds $ra_i \leq b_i \leq Ra_i$

2.1 Bounds for the largest eigen value of $L_{DPA}(G)$

Lemma A. Let G be regular graph with n vertices and m edges. Then the eigen values obtained from $L_{DPA}(G)$ matrix satisfies the following relations

$$(i) \sum_{i=1}^{n} \gamma_i = 2m \qquad (ii) \sum_{i=1}^{n} {\gamma_i}^2 = \sum_{i=1}^{n} {\alpha_i}^2 + M_1(G)$$

Where $\sum_{i=1}^{n} \alpha_i^2$ is the trace$(DPA(G))^2$ [9] and $M_1(G) = \sum_{i=1}^{n} (d_i)^2$, is the first Zagreb index [6].

Proof.

$$\text{(i)} \sum_{i=1}^{n} \gamma_i = trace[L_{DPA}(G)] = \sum_{i=1}^{n} d_i = n\, d_i = 2m$$

$$\text{(ii)} \sum_{i=1}^{n} (\gamma_i)^2 = trace[L_{DPA}(G)]^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} d_{ji} + \sum_{i=1}^{n} {d_i}^2, \quad \forall i \sim j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (d_{ij})^2 + \sum_{i=1}^{n} {d_i}^2$$

$$= 2 \sum_{1\leq i<j\leq n} (d_i \times d_j)^2 + \sum_{i=1}^{n} {d_i}^2$$

$$= \sum_{i=1}^{n} (\alpha_i)^2 + M_1(G)$$

Lemma B. Let β_i, $\forall i = 1, 2, ..., n$ be defined as above by Eq. (2). Then the following satisfies

$$(i) \sum_{i=1}^{n} \beta_i = 0 \qquad (ii) \sum_{i=1}^{n} {\beta_i}^2 = 2 \sum_{1\leq i<j\leq n} (d_i \times d_j)^2$$

$$= 2K$$

Where $K = \sum_{1 \le i < j \le n} (d_i \times d_j)^2$

Proof. Proof is similar to the proof of Lemma A.

Theorem 2.5. If G be a regular graph with n-vertices, then

$$\beta_1 \le \sqrt{\frac{2K(n-1)}{n}}$$

Proof. Consider the regular graph G with n-vertices. Let $L_{DPA}(G)$ be the Laplacian degree product adjacency matrix of graph G and $\beta_1, \beta_2, \beta_3, \ldots, \beta_n$ are the eigen values, where β_1 is the largest eigen value and the bound for β_1 is calculated by using cauchy-schwarz inequality i.e.,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right)$$

Let a_i =1 and $b_i = \alpha_i$, $\forall i$ = 2, 3, …, n then the inequality becomes,

$$\left(\sum_{i=2}^{n} (1)(\beta_i)\right)^2 \le \left(\sum_{i=2}^{n} 1^2\right) \cdot \left(\sum_{i=2}^{n} \beta_i^2\right)$$

(3)

From Lemma B(i),

$$\sum_{i=1}^{n} \beta_i = 0$$

$$\beta_1 + \sum_{i=2}^{n} \beta_i = 0$$

$$\left(\sum_{i=2}^{n} \beta_i\right)^2 = (-\beta_1)^2$$

(4)

And from Lemma B(ii)

$$\sum_{i=1}^{n} (\beta_i)^2 = 2K$$

$$(\beta_1)^2 + \sum_{i=2}^{n} (\beta_i)^2 = 2K$$

$$\sum_{i=2}^{n} (\beta_i)^2 = 2K - (\beta_1)^2$$

(5)

Substituting (2) and (3) in (1), we get

$$(-\beta_1)^2 \le (n-1)(2K - \beta_i^2)$$

$$\beta_1^2 \le 2K(n-1) - \beta_i^2(n-1)$$

$$\beta_1 \le \sqrt{\frac{2K(n-1)}{n}}$$

The equality relation for β_1 holds for all complete (K_n) graphs.

Theorem 2.6. If G be a regular graph with n-vertices, then

$$\sqrt{2K} \leq LE_{DPA}(G) \leq \sqrt{2nK}$$

Proof. Consider a regular graph G with n-vertices. Let $L_{DPA}(G)$ be the Laplacian degree product adjacency matrix of a graph G and $\beta_1, \beta_2, \beta_3, \ldots, \beta_n$ are the eigen values. Now we consider the cauchy-schwarz inequality to prove the theorem,

- **Proof for Right hand side bond:**

Let us assume that $a_i = 1$ and $b_i = |\beta_i|$, $\forall i = 1, 2, \ldots, n$.

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right).\left(\sum_{i=1}^{n} b_i^2\right)$$

$$\left(\sum_{i=1}^{n} (1)|\beta_i|\right)^2 \leq \left(\sum_{i=1}^{n} (1)^2\right).\left(\sum_{i=1}^{n} |\beta_i|^2\right)$$

$$\left(\sum_{i=1}^{n} |\beta_i|\right)^2 \leq n\left(\sum_{i=1}^{n} |\beta_i|^2\right)$$

On simplification, by using the Lemma B.

$$LE_{DPA}(G) \leq \sqrt{2nK} \qquad (6)$$

- **Proof for Left hand side bond:**

We know that,

$$\left(\sum_{i=1}^{n} |\beta_i|\right)^2 \geq \sum_{i=1}^{n} |\beta_i|^2$$

By using the Lemma B, we conclude that

$$LE_{DPA}(G) \geq \sqrt{2K} \qquad (7)$$

From equation (6) and (7),

$$\sqrt{2K} \leq LE_{DPA}(G) \leq \sqrt{2nK}$$

2.2 Lower bounds for the Laplacian degree product adjacency energy [$LE_{DPA}(G)$]

Theorem 2.7. Let G be a regular graph with n-vertices and m-edges and suppose $|\beta_1| \geq |\beta_2| \geq |\beta_3| \geq \ldots \geq |\beta_n|$ are the eigen values, then the following inequality holds.

$$LE_{DPA}(G) \geq \frac{2\sqrt{2nK|\beta_1||\beta_n|}}{|\beta_1| + |\beta_n|}$$

Proof. Consider a regular graph G with n-vertices and $|\beta_1| \geq |\beta_2| \geq |\beta_3| \geq \ldots \geq |\beta_n|$ are the eigen values, where $|\beta_1|$ and $|\beta_n|$ are the maximum and minimum eigen values of $|\beta_i|$'s respectively.

We have the inequality by the theorem 2.1,

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \le \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

Assume a_i =1, b_i=| β_i|, M_1M_2=| β_1| and m_1m_2=| β_n| then,

$$\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\beta_i|^2 \le \frac{1}{4}\left(\sqrt{\frac{|\beta_1|}{|\beta_n|}} + \sqrt{\frac{|\beta_n|}{|\beta_1|}}\right)^2 \left(\sum_{i=1}^{n} (1)|\beta_i|\right)^2$$

From Lemma B,

$$n2K \le \frac{1}{4}\left[\frac{(|\beta_1| + |\beta_n|)^2}{|\beta_1||\beta_n|}\right](LE_{DPA}(G))^2$$

$$(LE_{DPA}(G))^2 \ge \frac{8nK|\beta_1||\beta_n|}{(|\beta_1| + |\beta_n|)^2}$$

$$\mathrm{LE}_{DPA}(G) \ge \frac{2\sqrt{2nK|\beta_1||\beta_n|}}{|\beta_1| + |\beta_n|}$$

Theorem 2.8. Let G be a regular graph with n-vertices, then the following inequality holds

$$LE_{DPA}(G) \ge \sqrt{2nK - \frac{n^2}{4}(|\beta_1| - |\beta_n|)^2}$$

Proof. Consider a regular graph G with order n and size m. Let | β_1| ≥| β_2|≥| β_3|≥ ...≥| β_n| be the eigen values, where | β_1| and |β_n| are the maximum and minimum eigen values respectively.

From Theorem 2.2 we have the inequality,

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \frac{n^2}{4}(M_1M_2 - m_1m_2)^2$$

Assume a_i = 1, b_i = | β_i|, M_1M_2 =| β_1| and m_1m_2 = | β_n| in the above inequality,

$$\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\beta_i|^2 - \left(\sum_{i=1}^{n} (1)|\beta_i|\right)^2 \le \frac{n^2}{4}(|\beta_1| - |\beta_n|)^2$$

From Lemma B,

$$n2K - (LE_{DPA}(G))^2 \le \frac{n^2}{4}(|\beta_1| - |\beta_n|)^2$$

$$LE_{DPA}(G) \ge \sqrt{2nK - \frac{n^2}{4}(|\beta_1| - |\beta_n|)^2}$$

Theorem 2.9. Let G be a regular graph with n-vertices, then the following inequality holds.

$$LE_{DPA}(G) \ge \sqrt{2nK - \mu(n)(|\beta_1| - |\beta_n|)^2}$$

Proof. Consider a regular graph G with order n and size m. Let $|\beta_1| \geq |\beta_2| \geq |\beta_3| \geq \ldots \geq |\beta_n|$ be the eigen values, where $|\beta_1|$ and $|\beta_n|$ are the maximum and minimum eigen values respectively.

Consider inequality from the Theorem 2.3,

$$\left| n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \leq \mu(n)(A-a)(B-b)$$

Now assume that $a_i = b_i = |\beta_i|$, $A = B = |\beta_1|$ and $a = b = |\beta_n|$, then the inequality reduces to

$$\left| n\sum_{i=1}^{n} |\beta_i|^2 - \left(\sum_{i=1}^{n} |\beta_i| \right)^2 \right| \leq \mu(n)(|\beta_1| - |\beta_n|)(|\beta_1| - |\beta_n|)$$

From Lemma B,

$$\left| 2nK - (LE_{DPA}(G))^2 \right| \leq \mu(n)(|\beta_1| - |\beta_n|)^2$$

$$LE_{DPA}(G) \geq \sqrt{2nK - \mu(n)(|\beta_1| - |\beta_n|)^2}$$

Theorem 2.10. Let G be a regular graph with n-vertices, then the following inequality holds.

$$LE_{DPA}(G) \geq \frac{2K + n|\beta_1||\beta_n|}{|\beta_1| + |\beta_n|}$$

Proof. Consider a regular graph G with order n and size m. Let $|\beta_1| \geq |\beta_2| \geq |\beta_3| \geq \ldots \geq |\beta_n|$ be the eigen values, arranged in non-increasing order, where $|\beta_1|$ and $|\beta_n|$ are the maximum and minimum eigen values respectively.

We make use of the inequality from the Theorem 2.4,

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i^2 \leq (r+R)\left(\sum_{i=1}^{n} a_i b_i \right)$$

Assume $b_i = |\beta_i|$, $a_i = 1$, $r = |\beta_n|$ and $R = |\beta_1|$, then the inequality implies to

$$\sum_{i=1}^{n} |\beta_i|^2 + |\beta_n||\beta_1| \sum_{i=1}^{n} 1^2 \leq (|\beta_n| + |\beta_1|)\left(\sum_{i=1}^{n} (1)|\beta_i| \right)$$

From Lemma B,

$$2K + (|\beta_n||\beta_1|)(n) \leq (|\beta_n| + |\beta_1|) LE_{DPA}(G)$$

$$LE_{DPA}(G) \geq \frac{2K + |\beta_n||\beta_1|(n)}{|\beta_n| + |\beta_1|}$$

Conclusion: In this article, we compute the results on lower bounds for the Laplacian degree product adjacency energy. Further the eigen values of degree product adjacency matrix and laplacian degree product adjacency matrix are numerically same but they differ in their sign (i.e., positive and negative) and we consider the modulus to calculate the energy for both degree product adjacency matrix and Laplacian degree product adjacency matrix. Hence the energy become same for both $DPA(G)$ energy and $L_{DPA}(G)$ energy i.e., $E_{DPA}(G) = LE_{DPA}(G)$, for all regular graph G. Also the result is true for lower bounds.

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